3) Sequences of maps and normal families.

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous maps $f_n : X \to Y$ between two metric spaces. (suffices Y metric) If the sequence converges to some $f_\infty : X \to Y$ uniformly (on compact subsets of X), then the limit function f_∞ is also continuous. (Uniform limit theorem).

If X, Y are Riemann surfaces, and f_n are holomorphic, we have a similar statement.

Theorem (Weierstrass Uniform convergence thm).

If $(f_n)_{n\in\mathbb{N}}$ is a sequence of holomorphic functions between Riemann surfaces X, Y, which converges uniformly on compact subsets of X, then the limit f_∞ is holomorphic. Moreover, $f_n' \to f_\infty'$ uniformly on compact subsets.

Proof.

Being the statement local, we may assume $X = U \subseteq \mathbb{C}$ (for example $U = D(p, \delta)$)

Let $q = f_\infty(p)$, pick local coordinates at q, and set $V = D(q, r)$, $V' = D(q, \frac{r}{2})$

Replace U by $U' = f_\infty^{-1}(\underbrace{D(q, \tfrac{r}{2})}_{V'}) \cap U$, so that $f_\infty(U') \subseteq V'$.

open because f_∞ is continuous

By uniform convergence on compacts, $\exists N$ s.t. $\forall n > N$, $|f_n(z) - f_\infty(z)| < \frac{r}{2}$, so that $f_n(U') \subseteq V$.

We may assume that $f_n : U \to \mathbb{C}$.

Recall that $f : U \to \mathbb{C}$ continuous is holomorphic if and only if for any disc $D(p, r) \subset U$, $\gamma = \partial D(p, r)$, we have $\int_\gamma f(z)\,dz = 0$.

In this situation we have:

$$\left|\int_\gamma f_\infty(z)\,dz\right| \leq \left|\underbrace{\int_\gamma (f_\infty - f_n)(z)\,dz}_{\uparrow\ \text{compact support.}}\right| + \left|\underbrace{\int_\gamma f_n(z)\,dz}_{=0\ \text{because } f_n \text{ is holomorphic.}}\right| \leq \text{length}(\gamma)\cdot\varepsilon$$

Where $\forall \varepsilon > 0$ we picked $N \in \mathbb{N}$ so that $\forall n > N$, $|f_\infty(z) - f_n(z)| < \varepsilon$ $\forall z \in K = \text{Im}(\gamma)$

Letting ε to 0, we get $\int_\gamma f_\infty(z)\,dz = 0$, and f_∞ is holomorphic.

We now show that $f_n' \to f_\infty'$ uniformly on compact subsets.

Let $K \subset U$ be any compact set. By taking coverings by open discs, and taking finite subcoverings, we may assume $K = \overline{D(p, r)}$, with $r > 0$ such that $\overline{D(p, 2r)} \subset U$.

By uniform convergence, $\forall \varepsilon > 0$ $\exists N = N(\varepsilon)$ so that $|f_n(z) - f_\infty(z)| < \varepsilon$ $\forall z \in D(p, 2r)$. By Cauchy integral formula; $\forall z \in K = \overline{D(p, r)}$,

$$|f_n'(z) - f_\infty'(z)| = \frac{1}{2\pi}\left|\int_{|w-z|=s} \frac{f_n(w) - f_\infty(w)}{(w-z)^2}\,dw\right| \quad \text{for any } s \leq r.$$

$\nwarrow$ $w \in D(p, 2r)$.

But then, $\leq \frac{1}{2\pi}\cdot 2\pi s \cdot \frac{\varepsilon}{s^2} = \frac{\varepsilon}{s}$ as far as $n > N = N(\varepsilon)$

Hence $f_n' \to f_\infty'$ uniformly on K. $\square$

Topology of convergence on compact sets.

Given X, Y two Riemann surfaces (or more generally two locally compact topological spaces), we introduce a topology on $\mathcal{C}^0(X, Y)$, known as the "compact-open" topology.

Def: The compact-open topology is the weakest (coarsest) containing the sets $N_{K,U} = \{ f \in \mathrm{Map}(X,Y) \mid f(K) \subset U \}$ for any $K \overset{\subset X}{\text{compact}}$ and any $U \subset Y$ open.

Recall: the weakest topology is constructed as:

N open $\Leftrightarrow$ it is an arbitrary union of sets of the form $N_{K_1,U_1} \cap \dots \cap N_{K_n,U_n}$.

$\Leftrightarrow \forall f \in N$ $\exists$ $N_{K,U}$ s.t. $f \in N_{K,U} \subset N$.

When Y is a metric space, this topology is equivalent to the one generated by sets $N_{K,\varepsilon}(g) = \{ f \in \mathrm{Map}(X,Y) \mid d(f(x), g(x)) < \varepsilon \ \forall x \in K \}$.

$\supseteq$ $\forall N_{K,U}$, $g \in N_{K,U}$ take $\varepsilon \ll 1$ such that $\varepsilon < d(\underset{\text{closed}}{g(K)}, \underset{\text{closed}}{Y \setminus U}) \rightsquigarrow N_{K,\varepsilon}(g) \subset N_{K,U}$

since $f \in N_{K,\varepsilon}(g) \Rightarrow f(K) \subseteq \varepsilon$-nbhd of $g(K) \subseteq U$.

$\subseteq$ Finally, notice that given $\forall f \in N_{K,\varepsilon}(g)$, $\exists \eta > 0$ st $N_{K,\eta}(f) \subset N_{K,\varepsilon}(g)$.

In fact $K \ni x \mapsto d(f(x), g(x))$ is continuous so admits a max $\delta < \varepsilon$.

We can take $\eta = \varepsilon - \delta \rightsquigarrow \forall h \in N_{K,\eta}(f)$, $d(h(x), g(x)) \leq d(h(x), f(x)) + d(f(x), g(x)) < \eta + \delta < \varepsilon$.

Secondly, $\forall \varepsilon > 0$, $\forall x \in K$, pick V_x open in X such that $d(g(y), g(x)) < \varepsilon$ $\forall y \in V_x$ (by continuity of g).

So for any $y \in V_x$ we have $d(f(y), g(y)) \leq d(f(y), g(x)) + d(g(y), g(x))$ (3.4)

$\hat{\varepsilon}$

Set $U_x = D(g(x), \varepsilon)$, so that if $f(y) \in U_x$, then $d(f(y), g(y)) < 2\varepsilon$

Up to shrinking V_x, we may assume that $\exists K_x$ compact, $K_x \supset V_x$

Since K is compact, we can extract a finite covering $\{V_{x_1}, \dots V_{x_n}\}$ of $\{V_x\}$

Then if $f \in \bigcap_{j=1}^{n} \mathcal{N}_{K_{x_j}, U_{x_j}} \Rightarrow f \in \mathcal{N}_{K, 2\varepsilon}(g)$ □

Rem: If Y is a normed vector ~~field~~ space, $\mathcal{N}_{K,\varepsilon}(g) = \{ f \mid \| f|_K - g|_K \|_\infty < \varepsilon \}$

Lemma: $\mathcal{C}^0(X, Y)$ is a Hausdorff topological space. A sequence $(f_j)_{j \in \mathbb{N}}$ converges to $g \in \mathcal{C}^0(X, Y)$ in the compact-open topology if and only if:

(a) $\forall K \subset X$ compact, $f_j|_K : K \to Y$ converges uniformly to $g|_K$.

$\Leftrightarrow$ (b) f_j converges locally uniformly to g. ($\forall x \in X$ $\exists U$ nbhd of x, $f_j|_U \to g|_U$ unif.).

Proof: Automatic from the definitions

Rem: The compact-open topology does not depend on the distance picked in Y.

Now, if X, Y are Riemann surfaces, from Weierstrass theorem we deduce that $Hol(X, Y)$ is a closed subset of $\mathcal{C}^0(X, Y)$

Theorem (Hyperbolic compactness): If X and Y are hyperbolic Riemann surfaces, then $Hol(X, Y)$ is locally compact and σ-compact.

Moreover if $K_X \subset X$, $K_Y \subset Y$ are non-empty compact sets, then $\{ f \in H(X, Y) \mid f(K_X) \subseteq K_Y \}$ is compact

Lemma: $\mathcal{C}(X, Y)$ is metrisable (as long as X is σ-compact)

~~Proof~~: Rem: X is σ-compact (by definition) if it is covered by countably many compact sets.

$\hat{\mathbb{C}}$ is compact, $\mathbb{C}$ and $\mathbb{D}$ clearly satisfy this property. Since covering maps are proper, Any Riemann surface is σ-compact.

Proof (lemma). Since X is σ-compact and locally compact, $\exists$ sequence $K_1 \subset\subset K_2 \subset\subset K_3 \subset\subset \dots \subset\subset K_n \subset\subset \dots$ of nested compact sets so that $X = \bigcup_n K_n$

↑ I think $K_1 \subseteq K_2 \subseteq \dots \subseteq K_n \subseteq \dots$ suffices, no need for local compactly.

[$(H_n)_{n \in \mathbb{N}^*}$ given by σ-compactness. $K_1 = H_1$. Suppose we built K_n.

Take $H = K_n \cup H_{n+1}$, $\forall x \in H$, $\exists K_x$ cpct nbhd of x (by local compactness)
(cpct.)
$\Rightarrow K_x \supset U_x \ni x$
(open)

Extract finite covering $U_{x_1} \dots U_{x_J}$ $\Rightarrow$ $K_{n+1} = \bigcup_{j=1}^{J} K_{x_j}$ and $K_n \subset \bigcup_{j=1}^{J} U_{x_j}$.]

Denote by d the distance on Y, ~~...~~ Up to replacing d by $d \wedge 1$ $(\min\{d, 1\})$, we may assume $d(y, y') \leq 1$ $\forall y, y' \in Y$.

$\forall f, g \in \mathcal{C}(X, Y)$, set $D(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \max\{d(f(x), g(x)) \mid x \in K_n\}$.

D clearly defines a distance on $\mathcal{C}(f, g)$. We must show that its topology is the compact-open topology.

Let $B_D(f, \varepsilon)$ denote the open ball with respect to D, and $\mathcal{N}_{K,\varepsilon}(f)$ as above.

Fix f, ε. Let N be big enough so that $\frac{1}{2^N} < \frac{\varepsilon}{2}$. Set $K = K_N$. Then

$g \in \mathcal{N}_{K, \frac{\varepsilon}{2}}(f) \Rightarrow d(f(x), g(x)) < \frac{\varepsilon}{2}$ $\forall x \in K$ ($\supset K_n$ $\forall n \leq N$) $\Rightarrow D(f, g) \leq \frac{\varepsilon}{2} \cdot \underbrace{\sum_{n=1}^{N} \frac{1}{2^n}}_{\leq 1} + \sum_{n > N} \frac{1}{2^n} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

$\Rightarrow g \in B_D(f, \varepsilon)$

Conversely, given $\mathcal{N}_{K,\varepsilon}(f)$, let N big enough so that $K \subset K_N$.

In this case $B_D(f, \frac{\varepsilon}{2^N}) \subset \mathcal{N}_{K,\varepsilon}(f)$.

In fact: $\sum 2^{-n} d\max\{d(f(x), g(x)) \mid x \in K_n\} < \frac{\varepsilon}{2^N} \Rightarrow \max\{d(f(x), g(x)) \mid x \in K_N\} < \varepsilon$. □

Proof (of Theorem). Since $\mathcal{C}(X,Y)$, and hence $Hol(X,Y)$ is metrizable, by Bolzano-Weierstrass theorem, $\mathcal{F} \subset Hol(X,Y)$ is compact if and only if it is sequentially compact (every sequence admits a convergent subsequence).

Idea of Bolzano-Weierstrass: let $\mathcal{U} = (U_\alpha)_{\alpha \in A}$ be an open covering of F.

1) $\exists \rho > 0$ (called Lebesgue diameter of the covering) such that if $\operatorname{diam}(U) < \rho$ then $\exists \alpha \in A$, $U \subset U_\alpha$.

2) $\forall r > 0$ $\exists (B(x_j, r))_{j=1 \dots J(r)}$ finite covering.

3) Apply to $r < \rho$

Let $K_X \subset X$ and $K_Y \subset Y$ be two compacts and $\mathcal{F} = \{f \in Hol(X,Y) : f(K_X) \subset K_Y\}$

Let $\{x_j\}_{j \in \mathbb{N}^*}$ be a countable dense subset of X so that $\cancel{\#}\, x_1 \in K_X$.

(easy to construct for $\hat{\mathbb{C}}$, $\mathbb{C}$ or $\mathbb{D}$, just take points with coordinates in $\mathbb{Q}$ }, then project)

$\forall f \in \mathcal{F}$, $f(x_1) \in K_Y$, hence given a sequence $f_n \in \mathcal{F}$, we can extract a subsequence f_{n_j} so that $f_{n_j}(x_1) \to y_1 \in K_Y$. Set $Q_1 = (n_j)$ the indices.

We claim that $\{f_{n_j}(x_2)\}$ belong to a compact set in Y.

We first show that $(f_{n_j}(x_2))$ is a Cauchy sequence

3.7

Since $f_{n_j}(x_1) \to y_1$, $(f_{n_j}(x_1))$ is Cauchy, and $\forall \varepsilon > 0$ $\exists J = J_\varepsilon \in \mathbb{N}$ so that $\forall j_1, j_2 \geq J$, $d_Y(f_{n_{j_1}}(x_1), f_{n_{j_2}}(x_1)) < \varepsilon$.

Then $d_Y(f_{n_{j_1}}(x_2); f_{n_{j_2}}(x_2)) \leq d_Y(f_{n_{j_1}}(x_2), f_{n_{j_1}}(x_1)) + d_Y(f_{n_{j_1}}(x_1), f_{n_{j_2}}(x_1)) +$
$+ d_Y(f_{n_{j_2}}(x_1), f_{n_{j_2}}(x_2)) \leq 2 d_X(x_1, x_2) + \varepsilon$.

In particular $f_{n_j}(x_2) \in \overline{B_Y(f_{n_J}(x_2); 2d_X(x_1,x_2) + \varepsilon)} \cup \{f_{n_k}(x_2) \mid k < J\}$, which is compact.

We can extract a subsequence $Q_2 = \{(n_{j_k})\}$ so that $f_{n_{j_k}}(x_2) \to y_2 \in Y$.

We write: $\lim\limits_{\substack{n \in Q_2 \\ n \to \infty}} f_n(x_2) \to y_2$ for simplicity.

By recursion, we construct $Q_1 \supset Q_2 \supset \dots \supset Q_k \supset$, so that $(f_n(x_k))_{n \in Q_k}$ converges to some y_k.

Using a diagonal procedure, we construct a sequence of indices $\hat{Q}$ by taking the first element of Q_1, the second of Q_2, the third of Q_3, etc.

Then $\lim\limits_{\substack{n \in \hat{Q} \\ n \to \infty}} f_n(x_k) = y_k \quad \forall k \in \mathbb{N}^*$.

We claim that the sequence $(f_n)_{n \in \hat{Q}}$ converges uniformly on compacts to a limit function $f_\infty : X \to Y$, which will be holomorphic by Weierstrass uniform convergence theorem 2. ~~Clearly~~ ~~$f_\infty(x_k) = y_k$~~

Let L be any compact in X. $\forall \varepsilon > 0$ fixed, we can cover L by finitely many open d_X-balls of center x_j. Call I this finite set of indices.

Moreover, $\exists N$ s.t. $d_Y(f_n(x_j), f_m(x_j)) < \varepsilon$ $\forall j \in I$, $n,m \in \hat{Q}$, $n,m \geq N$. 3.8

(Since $(f_n(x_j))_{n\in\hat{Q}}$ converges, and there are only finitely many).

$\Rightarrow$ $\forall z \in L$, we have: $\exists j \in I$ s.t. $d_X(z, x_j) < \varepsilon$, and:

$$d_Y(f_n(z), f_m(z)) \leq \underbrace{d_Y(f_n(z), f_n(x_j))}_{\overset{\wedge}{d_X(z,x_j)}} + \underbrace{d_Y(f_n(x_j), f_m(x_j))}_{\overset{\wedge}{\varepsilon}} + \underbrace{d_Y(f_m(x_j), f_m(z))}_{\overset{\wedge}{d_X(z,x_j)}}$$

$\leq 3\varepsilon$. $\quad (\forall n,m \geq N,\ n,m \in \hat{Q})$

$\Rightarrow$ $(f_n(z))_{n\in\hat{Q}}$ is a d_Y-Cauchy sequence in Y, hence it converges (in K_Y)

Since N depends only on ε and L (and not z), the convergence is uniform, and $(f_n)_{n\in\hat{Q}}$ converges uniformly on compact sets to a limit function $f_\infty \in \mathcal{F}$.

~~Consider~~ Fix any point $x \in X$, and consider the evaluation map $ev_x : \mathrm{Hol}(X,Y) \to Y$, $f \mapsto f(x)$.

Then ev_x is continuous and proper by what we saw ($K_X = \{x\}$ is compact).

Since Y is locally compact and σ-compact, so is $\mathrm{Hol}(X,Y)$. $\square$

Rem: The statement is false in the non hyperbolic case.

Consider $f_n(z) = nz$ in $\hat{\mathbb{C}}$, $\mathbb{C}$, $\mathbb{C}/\mathbb{Z}$ or $\mathbb{C}/\Lambda$. $(X = Y =$ one of these$)$

Consider $K_X = K_Y = \{0\}$. then $f_n(K_X) \subset K_Y$. but f_n does not have any convergent subsequence, since $f_n'(0) = n \to \infty$.

$\mathrm{Hol}(\hat{\mathbb{C}}, \hat{\mathbb{C}})$ is locally compact but not compact. ($\mathrm{Hol}_d(\hat{\mathbb{C}}, \hat{\mathbb{C}})$ is a non-cpct $(2d+1)$-dim. manifold)

$\mathrm{Hol}(\mathbb{C}, \mathbb{C})$ is not locally compact

$f_n(z) = \varepsilon(1 + \varepsilon z + \dots + \varepsilon^n z^n)$ has no limit point in $\mathrm{Hol}(\mathbb{C}, \mathbb{C})$.

$\varepsilon \in B_{\mathbb{D}}(0, \hat{z})$.